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The orthogonality of the fractional circle polynomials and its application in modeling of ophthalmic surfaces

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Abstract. In this paper we establish some new fractional differential properties for a class of fractional circle polynomials. We apply the Zernike polynomials and a new class of fractional circle polynomials in modeling ophthalmic surfaces in visual optics and we compare the obtained results. The total RMS error is presented when addressing capability of these functions in fitting with surfaces, and it is showed that the new fractional circle polynomials can be used as an alternative to the Zernike Polynomials to represent the complete anterior corneal surface.

1. Introduction

The ocular aberrations are commonly described in terms of series of Zernike polynomials that offer distinct advances due to their normalization on a circular pupil, however, in some cases with slow convergence they may not be the most appropriate choice. This fact was analyzed in [1], where the authors showed a better fitting accuracy of circular Bessel Functions for abrupt variations of post-surgical corneal surfaces. The Bessel functions were also used in [2]. Here the authors studied the of a Bessel beacon generated with a spatial light modulator as a fixation target for ophthalmic adaptive optics systems, rather than a conventional point-spread-function. In [2] it was showed an evidence of an increased immunity for defocus fluctuations and was examined the power spectral density variations of the individual Zernike terms.

In this work, we consider a class of fractional circle polynomials which has orthogonal sets on a circular pupil, and we establish some new fractional differential properties for this class of fractional circle polynomials. We aim to investigate the applicability of this class of special functions in modelling ophthalmic surfaces. Moreover, we compare our results with the correspondent ones obtained when Zernike polynomials are used.

The structure of the paper reads as follows: in the Preliminaries section, we recall some basic facts about fractional calculus and the g-Jacobi functions. In the 3rd section, we introduce a new class of fractional circle polynomials and we study some properties of these polynomials, namely, orthogonality relations and fractional differential properties. In the last section, we apply the introduced class of polynomials to the modulations of ophthalmic surfaces, and we compare the fitting properties with those presented by the Zernike polynomials.

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2. Fractional calculus and g-Jacobi functions

We start by recalling the definition of fractional derivatives and fractional integrals in the sense of Riemann-Liouville (for more details see [3]).

Definition 2.1 *For* $t > 0$

$$J^{\nu} f(t) := \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} f(\tau)\, d\tau$$

is called the Riemann-Liouville fractional integral of the function $f(t)$ *of order* ν *with* $\mathrm{Re}(\nu) > 0$.

Definition 2.2 *If* $t > 0$ *and* $m \in \mathbb{N}_0$ *such that* $m-1 \leq \nu < m$*, then the fractional derivative of* $f(t)$ *of order* ν *is defined as*

$$D^{\nu} f(t) = D^m \left[J^{m-\nu} f(t) \right], \tag{1}$$

(if it exists) where $m - \nu > 0$.

We now turn to the definition and basic properties of the g-Jacobi functions introduced and studied in [4]. These functions play a fundamental role in defining the fractional Zernike polynomials, and correspond to a generalization for the fractional case of classical Jacobi polynomials.

Definition 2.3 *We define the (generalized or) g-Jacobi functions by the formula*

$$P_{\nu}^{(\alpha,\beta)}(t) = (-2)^{-\nu}\, \Gamma(\nu+1)^{-1}\, (1-t)^{-\alpha}\, D^{\nu} \left[(1-t)^{\nu+\alpha} (1+t)^{\nu+\beta} \right], \qquad \nu > 0, \tag{2}$$

where $\alpha > -1$, $\beta > -1$ *and* D^{ν} *is the Riemann-Liouville fractional differential operator (1).*

In the following results, we recall some properties of the g-Jacobi functions (see [4]), which are analogous to the corresponding properties of the classical Jacobi polynomials. In the first two results, we provide the corresponding explicit formulas for these functions.

Theorem 2.4 *For the g-Jacobi functions holds the explicit formula*

$$P_{\nu}^{(\alpha,\beta)}(t) = 2^{-\nu} \sum_{k=0}^{\infty} \left(\begin{array}{c} \nu+\alpha \\ \nu-k \end{array} \right) \left(\begin{array}{c} \nu+\beta \\ k \end{array} \right) (t-1)^k (t+1)^{\nu-k}, \tag{3}$$

where

$$\left(\begin{array}{c} \alpha \\ \beta \end{array} \right) = \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)\,\Gamma(1+\alpha-\beta)} \tag{4}$$

is the binomial coefficient with real arguments.

Theorem 2.5 *The g-Jacobian functions can be represented as*

$$P_{\nu}^{(\alpha,\beta)}(t) = \left(\begin{array}{c} \nu+\alpha \\ \nu \end{array} \right) {}_2F_1 \left(-\nu,\, \nu+\alpha+\beta;\, \alpha+1;\, \frac{1-t}{2} \right) \tag{5}$$

$$= \frac{1}{\Gamma(1+\nu)} \sum_{k=0}^{\infty} \left(\begin{array}{c} \nu \\ k \end{array} \right) \frac{\Gamma(1+\nu+\alpha+\beta+k)\,\Gamma(1+\alpha+\nu)}{\Gamma(1+\nu+\alpha+\beta)\,\Gamma(1+\alpha+k)} \left(\frac{t-1}{2} \right)^k, \tag{6}$$

where ${}_2F_1(a,\, b;\, c;\, t)$ *is the Gauss hypergeometric function.*

3. Fractional circle polynomials

The aim of this section is to introduce a new class of fractional circle polynomials and to study some of their main properties. Taking into account the definition of g-Jacobi functions presented previously and the ideas presented in [5], we introduce the definition for a new class of fractional circle polynomials.

Definition 3.1 *We define the fractional circle polynomials by*

$$W_\nu^{-m}(\rho,\theta) = R_\nu^m(\rho)\,\sin(m\theta), \qquad \text{when } [\nu-m] \text{ is odd }, \tag{7}$$

$$W_\nu^{m}(\rho,\theta) = R_\nu^m(\rho)\,\cos(m\theta), \qquad \text{when } [\nu-m] \text{ is even }, \tag{8}$$

where $m\in\mathbb{N}_0$, $\nu\in\mathbb{R}_+$, $\nu>m$, $0\le\rho\le 1$ is the radial distance, $0\le\theta\le 2\pi$ is the azimuthal angle, and $[\beta]$ represents the integer part of β. The fractional radial function $R_\nu^m(\rho)$ will be defined as

$$R_\nu^m(\rho) = (-1)^{\left[\frac{\nu-m}{2}\right]}\,\rho^m\,P_{\frac{\nu-m}{2}}^{\left(m,\frac{1}{2}\right)}(1-2\rho^2), \tag{9}$$

where $P_\nu^{(\alpha,\beta)}(x)$ are the g-Jacobi functions defined in (2).

In [6] it was proved the following orthogonality relation for the g-Jacobi polynomials $P_\nu^{\alpha,\beta}(x)$:

Lemma 3.2 *(cf. [6]) The following relation holds*

$$\int_{-1}^{1}(1-x)^\alpha\,(1+x)^\beta\,P_{n+\theta}^{(\alpha,\beta)}(x)\,P_{m+\theta}^{(\alpha,\beta)}(x)\,dx = \psi\,(n,\theta,\alpha,\beta)\,\delta_{n,m},$$

where $\delta_{n,m}$ denotes the Kronecker number and

$$\psi(n,\theta,\alpha,\beta) = \frac{(-1)^\theta 2^{\alpha+\beta+1}\Gamma(\alpha+n+\theta+1)\Gamma(\beta+n+\theta+1)}{\Gamma(1-\theta)\Gamma(1+\theta)(2n+\alpha+\beta+\theta+1)\Gamma(n+\alpha+\beta+\theta+1)\Gamma(n+1+\theta)},$$

for $0\le\theta<1$, $\alpha>-1$ and $\beta>-1$.

It is also possible to obtain an orthogonality relation for the fractional circle polynomials $W_\nu^m(\rho,\theta)$, because their radial part is written in terms of the orthogonal class of g-Jacobi polynomial $P_\nu^{\alpha,\beta}(x)$. The figure below show the pyramid with the first 15 modes of $W_{n+\frac{1}{2}}^m$.

Now, we establish a fractional differential property satisfied by the fractional circle polynomials $W_{n+\frac{1}{2}}^m$. In order to do that, we initially proved a fractional differential property for the radial part of $W_{n+\frac{1}{2}}^m$, i.e., for $R_\nu^m(\rho)$.

Theorem 3.3 *The fractional radial function $R_\nu^m(\rho)$ satisfies the following fractional partial differential equation*

$$D_\rho^\alpha R_\nu^m(\rho) + \frac{\rho}{\alpha+1}\,D_\rho^{\alpha+1}R_\nu^m(\rho) = \frac{\Gamma(1+m)\rho^{m-\alpha}}{(\alpha+1)\Gamma(m-\alpha)}\,{}_2F_1\left(\frac{m-\nu}{2},\frac{m+\nu+1}{2};\,m+1;\,\rho^2\right),$$

where D_ρ^α is defined via (1).

Proof: Taking into account (9) and (5) we have

$$\begin{aligned} D_\rho^\alpha R_\nu^m(\rho) &= (-1)^{\left[\frac{\nu-m}{2}\right]}\,D_\rho^\alpha\left[\rho^m\,P_{\frac{\nu-m}{2}}^{\left(m,\frac{1}{2}\right)}(1-2\rho^2)\right] \\ &= (-1)^{\left[\frac{\nu-m}{2}\right]}\begin{pmatrix}\frac{\nu+m}{2}\\ \frac{\nu-m}{2}\end{pmatrix}D_\rho^\alpha\left[\rho^m\,{}_2F_1\left(\frac{m-\nu}{2},\frac{m+\nu+1}{2};\,m+1;\,\rho^2\right)\right]. \end{aligned}$$

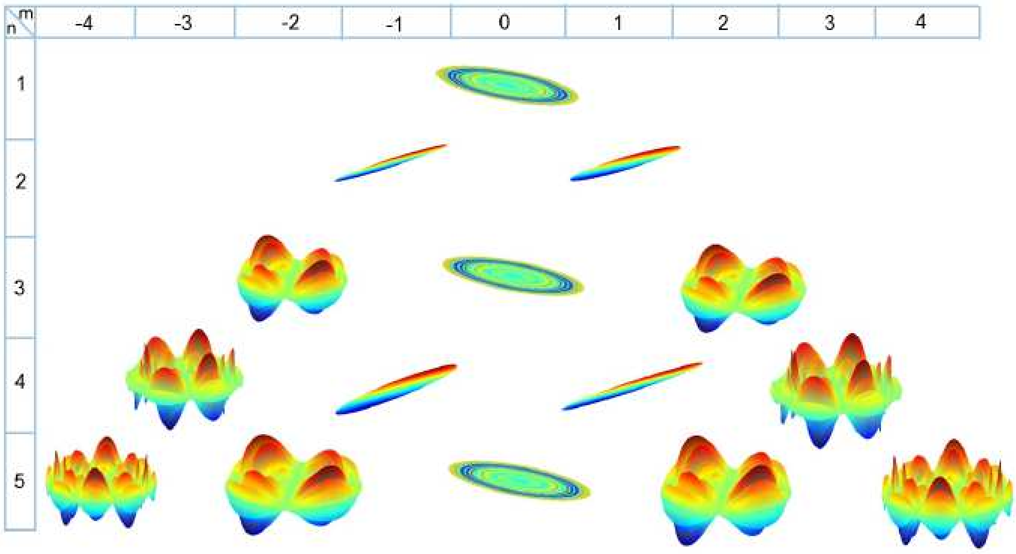

Figure 1. Pyramid for W.

Then, we derive the explicit form for the fractional partial derivative that appears in the right hand side of the last expression. For that, we apply the Leibnitz product rule presented in [7], and after straightforward calculation we obtain

$$
\begin{aligned}
& D_\rho^\alpha\left[\rho^m\,{}_2F_1\left(\frac{m-\nu}{2},\frac{m+\nu+1}{2};\,m+1;\,\rho^2\right)\right] \\
&= \sum_{j=0}^{\infty}\left(\begin{array}{c}\alpha\\ j\end{array}\right) D_\rho^{\alpha-j}\rho^m\; D_\rho^j\left[{}_2F_1\left(\frac{m-\nu}{2},\frac{m+\nu+1}{2};\,m+1;\,\rho^2\right)\right] \\
&= \Gamma(1+m)\,\rho^{m-\alpha}\sum_{j=0}^{\infty}\left(\begin{array}{c}j-\alpha-1\\ j\end{array}\right)\frac{(-\rho)^j}{\Gamma(1+m-\alpha+j)}\,D_\rho^j\left[{}_2F_1\left(\frac{m-\nu}{2},\frac{m+\nu+1}{2};\,m+1;\,\rho^2\right)\right] \\
&= \frac{\Gamma(1+m)}{\Gamma(1+m-\alpha)}\,\rho^{m-\alpha}\sum_{j=0}^{\infty}\frac{\Gamma(j-\alpha)\,\Gamma(1+m-\alpha)\,(-\rho)^j}{\Gamma(-\alpha)\,\Gamma(1+m-\alpha+j)\,j!}\,D_\rho^j\left[{}_2F_1\left(\frac{m-\nu}{2},\frac{m+\nu+1}{2};\,m+1;\,\rho^2\right)\right] \\
&= \frac{\Gamma(1+m)\,(m-\alpha)}{\Gamma(1+m-\alpha)\,(-\alpha-1)}\,\rho^{m-\alpha} \\
&\quad\times\sum_{j=0}^{\infty}\frac{\Gamma(j-\alpha)\,\Gamma(m-\alpha)\,(-\rho)^j}{\Gamma(-\alpha-1)\,\Gamma(1+m-\alpha+j)\,j!}\,D_\rho^j\left[{}_2F_1\left(\frac{m-\nu}{2},\frac{m+\nu+1}{2};\,m+1;\,\rho^2\right)\right] \\
&= \frac{\Gamma(1+m)\,(m-\alpha)}{(m-\alpha)\Gamma(m-\alpha)\,(-\alpha-1)}\,\rho^{m-\alpha} \\
&\quad\times\sum_{j=0}^{\infty}\frac{\Gamma(j-\alpha)\,\Gamma(m-\alpha)\,(-\rho)^j}{\Gamma(-\alpha-1)\,\Gamma(1+m-\alpha+j)\,j!}\,D_\rho^j\left[{}_2F_1\left(\frac{m-\nu}{2},\frac{m+\nu+1}{2};\,m+1;\,\rho^2\right)\right]
\end{aligned}
$$

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$$= \frac{\Gamma(1+m)}{\Gamma(m-\alpha)\,(-\alpha-1)}\,\rho^{m-\alpha}$$
$$\times \sum_{j=0}^{\infty} \frac{\Gamma(j-\alpha)\,\Gamma(m-\alpha)\,(-\rho)^j}{\Gamma(-\alpha-1)\,\Gamma(1+m-\alpha+j)\,j!}\, D_\rho^j \left[{}_2F_1\left(\frac{m-\nu}{2}, \frac{m+\nu+1}{2}; m+1; \rho^2\right)\right]. \tag{10}$$

Analogously, we deduce

$$D_\rho^{\alpha+1}\left[\rho^m\, {}_2F_1\left(\frac{m-\nu}{2}, \frac{m+\nu+1}{2}; m+1; \rho^2\right)\right]$$
$$= \frac{\Gamma(1+m)}{\Gamma(m-\alpha)}\,\rho^{m-\alpha-1} \sum_{j=0}^{\infty} \frac{\Gamma(j-\alpha-1)\,\Gamma(m-\alpha)\,(-\rho)^j}{\Gamma(-\alpha-1)\,\Gamma(m-\alpha+j)\,j!}\, D_\rho^j \left[{}_2F_1\left(\frac{m-\nu}{2}, \frac{m+\nu+1}{2}; m+1; \rho^2\right)\right]. \tag{11}$$

So,

$$D_\rho^{\alpha}\left[\rho^m\, {}_2F_1\left(\frac{m-\nu}{2}, \frac{m+\nu+1}{2}; m+1; \rho^2\right)\right] - \frac{\rho}{-\alpha-1} D_\rho^{\alpha+1}\left[\rho^m\, {}_2F_1\left(\frac{m-\nu}{2}, \frac{m+\nu+1}{2}; m+1; \rho^2\right)\right]$$
$$= \frac{\Gamma(1+m)\rho^{m-\alpha}}{(-\alpha-1)\Gamma(m-\alpha)}$$
$$\times \sum_{j=0}^{\infty} \left(\frac{\Gamma(j-\alpha)}{\Gamma(1+m-\alpha+j)} - \frac{\Gamma(j-\alpha-1)}{\Gamma(m-\alpha+j)}\right) \frac{\Gamma(m-\alpha)\,(-\rho)^j}{\Gamma(-\alpha-1)j!}\, D_\rho^j \left[{}_2F_1\left(\frac{m-\nu}{2}, \frac{m+\nu+1}{2}; m+1; \rho^2\right)\right]. \tag{12}$$

The series present in (12) can be looked as a convergent Mengoli's series, and we have

$$\sum_{j=0}^{\infty} \left(\frac{\Gamma(j-\alpha)}{\Gamma(1+m-\alpha+j)} - \frac{\Gamma(j-\alpha-1)}{\Gamma(m-\alpha+j)}\right) \frac{\Gamma(m-\alpha)\,(-\rho)^j}{\Gamma(-\alpha-1)j!}\, D_\rho^j \left[{}_2F_1\left(\frac{m-\nu}{2}, \frac{m+\nu+1}{2}; m+1; \rho^2\right)\right]$$
$$= -{}_2F_1\left(\frac{m-\nu}{2}, \frac{m+\nu+1}{2}; m+1; \rho^2\right).$$

■

The previous theorem immediately implies the following corollary.

Corollary 3.4 *The fractional circle polynomials $W_\nu^{\pm m}(\rho,\theta)$ satisfy the following fractional partial differential equation*

$$D_\rho^{\alpha} W_\nu^{\pm m}(\rho,\theta) + \frac{\rho}{\alpha+1} D_\rho^{\alpha+1} W_\nu^{\pm m}(\rho,\theta) = \frac{\Gamma(1+m)\rho^{m-\alpha}}{(\alpha+1)\Gamma(m-\alpha)}\, {}_2F_1\left(\frac{m-\nu}{2}, \frac{m+\nu+1}{2}; m+1; \rho^2\right).$$

4. Application in modeling of ophthalmic surfaces

In this section, we compare the behaviour of introduced fractional circle polynomials with the Zernike circle polynomials when both are applied in the modulation of ophthalmic surfaces. The evaluation of the RMS error in fitting these polynomials with the specific surfaces is chosen as an effective indicator for the comparison of these polynomials. In each case, we use 15 polynomials of the orthogonal set. The surface of interest can be modeled by:

$$C(r,\phi) = \sum_{p=1}^{P} a_p \psi_p(r,\phi) + \varepsilon_p(r,\phi), \tag{13}$$

where the index p is a polynomial ordering-number, $\psi_p(r,\phi)$, with $p = 1,\ldots,P$, is the pth polynomial, a_p, with $p = 1,\ldots,P$, is the coefficient associated with $\psi_p(r,\phi)$, P is the order, r is the normalised distance from the origin, ϕ is the angle and $\varepsilon_p(r,\phi)$ represents the modeling error. Throughout this analysis, we chose the polar coordinate system for convenience. Using a set of such discrete orthogonal polynomials, we can form a linear model

$$C = \psi a + \varepsilon, \tag{14}$$

where C is a $D-$element vector column of surface evaluated at discrete points (r_d, ϕ_d), with $d = 1,\ldots,D$, ψ is a $(D \times P)$ matrix of discrete orthogonal polynomials $\psi_p(r_d,\phi_d)$, a is a $P-$element vector column of coefficients, and ε represents a D-element vector column of the measurement and modeling error. For this model, the coefficient vector a can be estimated using the least squares method, that is

$$\widehat{a} = (\psi^T\psi)^{-1}\psi^T C, \tag{15}$$

where T denotes the transposition since the inverse exists. The RMS error can be found by

$$RMS_{error} = \frac{\sqrt{\sum_{p=1}^{P}(\varepsilon_p(r,\phi))^2}}{P}. \tag{16}$$

For Zernike's polynomials, the most widely used representation of its radial function is (see [5]):

$$R_n^{|m|}(r) = \sum_{s=0}^{(n-|m|)/2} \frac{(-1)^s(n-s)!}{s!(\frac{n-|m|}{2}-s)!(\frac{n+|m|}{2}-s)!} r^{n-2s}, \tag{17}$$

where $n \geq m$ so the parity of a polynomial is the same as the corresponding n. The Zernike polynomials are given in the normalized form by:

$$Z_n^m(r,\phi) = R_n^m(r)\cos(m\phi), \text{ for even m}$$
$$Z_n^{-m}(r,\phi) = R_n^m(r)\sin(m\phi), \text{ for odd m}. \tag{18}$$

In the case of the fractional circle polynomials on a circular disk, they are given by

$$W_{n+\frac{1}{2}}^m(r,\phi) = r^{|m|}P_{n+\frac{1}{2}}^{(\alpha,\beta)}(2r^2-1)\exp(mi\phi). \tag{19}$$

In order to evaluate the RMS error in fitting these polynomials with the specific surfaces, we selected 15 modes of each orthogonal set.

- Arrangement of type 1: In the case of the Zernike circle polynomials, from the standard pyramid we consider the first 15 modes to our analysis ($m = -4,-3,\ldots,3,4$ and $n = 0,1,2,3,4$). For the introduced fractional circle polynomials we consider $\beta = 1/2$, $\alpha = 1$, $m = 1$ and $n = 1,\ldots,15$.
- Arrangement of type 2: In the case of the Zernike circle polynomials, from the standard pyramid we consider the first 15 modes. Concerning the fractional circle polynomials, we consider the first 15 modes of the pyramid presented in Figure 1.

In our analysis, we have considered an interesting case that is the model of the total anterior eye surface (Fgure 2), which includes the anterior surface of the cornea, limbus and sclera. This was done by etching together two spherical surfaces of different radius. To produce this model we employed typical parameters for anterior corneal radius, visible iris and diameter of the eye. As shown in table below, the smallest error is given by the fractional circle polynomials in the case of arrangement of type 1. On the other hand, for arrangement of type 2 the Zernike polynomials present the smallest fitting error.

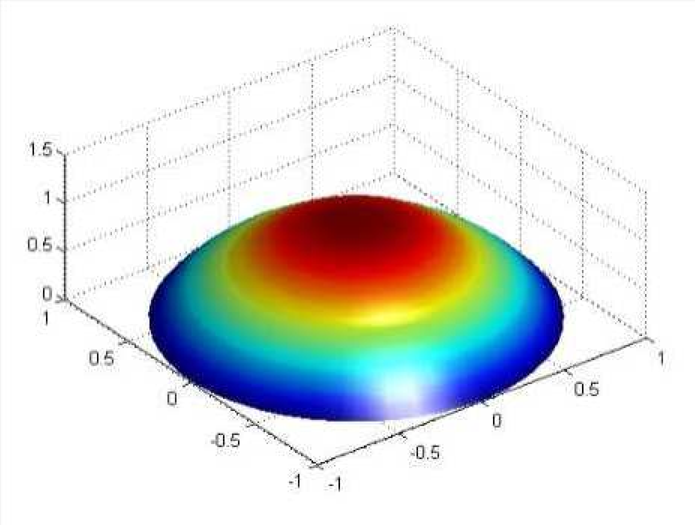

Figure 2. Total eye model with typical parameters.

	Arrangement of type 1	Arrangement of type 2
Zernike Polynomials	$9.49\,\mu m$	$9.49\,\mu m$
Fractional Circle Polynomials	$2.9\,\mu m$	$142\,\mu m$

Based on these examples, it is clear that in the case of the fractional circle polynomials the models with more irregularities lead to an increase of the RMS error.

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